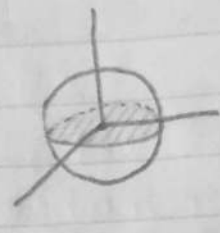
10/22 Calc III lecture notes

more double integrals:

Q: what is the volume of a sphere of radius $\alpha > 0$

A: (using what we know)

graphic:



$x^2 + y^2 + z^2 = \alpha^2$

$Vol(S_\alpha) = \iint_{R_\alpha} h(x,y)\, dA$

if we solve "$x^2 + y^2 + z^2 = \alpha^2$" for z

we obtain:

upper hemisphere $\rightarrow$ $z = \sqrt{\alpha^2 - x^2 - y^2}$

lower hemisphere $\rightarrow$ $z = -\sqrt{\alpha^2 - x^2 - y^2}$

height = upper hemisphere − lower hemisphere

$h(x,y) = 2\sqrt{\alpha^2 - x^2 - y^2}$

region of integration: $R = \{(x,y) : x^2 + y^2 \leq \alpha^2\}$

the upper semicircle of boundary R_α is

$y = \sqrt{\alpha^2 - x^2}$

the lower semicircle of boundary R_α is

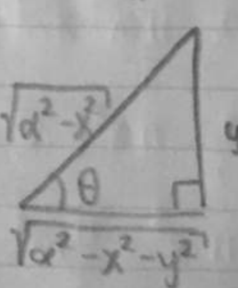
$y = -\sqrt{\alpha^2 - x^2}$

$\therefore R_\alpha = \{(x,y) : -\alpha \leq x \leq \alpha, -\sqrt{\alpha^2 - x^2} \leq y \leq \sqrt{\alpha^2 - x^2}$

hence, $Vol(S_\alpha) = \int_{-\alpha}^{\alpha} \int_{-\sqrt{\alpha^2 - x^2}}^{\sqrt{\alpha^2 - x^2}} 2\sqrt{\alpha^2 - x^2 - y^2}\, dy\, dx$

inner integral: $\int_{-\sqrt{\alpha^2 - x^2}}^{\sqrt{\alpha^2 - x^2}} 2\sqrt{\alpha^2 - x^2 - y^2}\, dy$

graphic:



(inner integral continued)

$$\sin(\theta) = y/\sqrt{\alpha^2 - x^2}$$
$$y = \sqrt{\alpha^2 - x^2}\,\sin(\theta)$$
$$dy = \sqrt{\alpha^2 - x^2}\,\cos(\theta)\,d\theta$$
$$\sqrt{\alpha^2 - x^2 - y^2} = \sqrt{\alpha^2 - x^2}\,\cos(\theta)$$

(lets ignore the bounds for a while)

$$\int 2\sqrt{\alpha^2 - x^2 - y^2}\,dy = 2\int \sqrt{\alpha^2 - x^2}\cos(\theta)\sqrt{\alpha^2 - x^2}\cos(\theta)\,d\theta$$
$$= 2(\alpha^2 - x^2)\int \cos^2(\theta)\,d\theta$$

recall: $\cos^2(\alpha) = 1/2\,(1 + \cos(2\alpha))$

$$= (\alpha^2 - x^2)\int 1 + \cos(2\theta)\,d\theta$$

$2\theta = w \quad 2d\theta = dw \quad d\theta = \frac{1}{2}dw$

$$= (\alpha^2 - x^2)\left(\theta + \frac{1}{2}\sin(2\theta)\right) + c$$

recall: $\sin(2\theta) = 2\sin\theta\cos\theta$

$$= (\alpha^2 - x^2)(\theta + \sin\theta\cos\theta) + c$$
$$= (\alpha^2 - x^2)\left(\arcsin\left(y/\sqrt{\alpha^2 - x^2}\right) + \left(y/\sqrt{\alpha^2 - x^2}\right)\left(\sqrt{\alpha^2 - x^2 - y^2}/\sqrt{\alpha^2 - x^2}\right)\right) + C$$
$$= (\alpha^2 - x^2)\arcsin\left(y/\sqrt{\alpha^2 - x^2}\right) + y\sqrt{\alpha^2 - x^2 - y^2} + C$$

$$\therefore \int_{-\sqrt{\alpha^2 - x^2}}^{\sqrt{\alpha^2 - x^2}} 2\sqrt{\alpha^2 - x^2 - y^2}\,dy = (\alpha^2 - x^2)\arcsin\left(y/\sqrt{\alpha^2 - x^2}\right) + y\sqrt{\alpha^2 - x^2 - y^2}\,\Bigg|_{-\sqrt{\alpha^2 - x^2}}^{\sqrt{\alpha^2 - x^2}}$$

$$(\alpha^2 - x^2)\arcsin(1) + \sqrt{\alpha^2 - x^2}\sqrt{0} - (\alpha^2 - x^2)\arcsin(-1) + \sqrt{\alpha^2 - x^2}\sqrt{0}$$
$$(\alpha^2 - x^2)(\arcsin(1) - \arcsin(-1)) = (\alpha^2 - x^2)(\pi/2 + \pi/2)$$
$$= (\alpha^2 - x^2)\pi$$

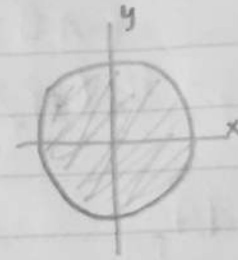
outer integral: finally!

$$\int_{-\alpha}^{\alpha} \pi(\alpha^2 - x^2)\,dx = \pi\alpha^2 x - \frac{\pi}{3}x^3\,\Bigg|_{-\alpha}^{\alpha}$$
$$= \pi\alpha^2\alpha - \frac{\pi}{3}\alpha^3 - \pi\alpha^2(-\alpha) + \frac{\pi}{3}(-\alpha)^3$$
$$= \pi\alpha^3 - \frac{\pi}{3}\alpha^3 + \pi\alpha^3 - \frac{\pi}{3}\alpha^3$$
$$= 4\pi\alpha^3 = \mathrm{vol}(S_\alpha) \quad \boxtimes$$

This computation was complicated; so it seems it would be more natural to use polar coordinates to describe the region and the height function.

the region:

cartesian →

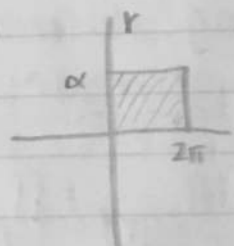


$x^2 + y^2 \leq \alpha^2$ [cartesian]

$0 \leq r \leq \alpha$ [polar]

$0 \leq \theta \leq 2\pi$

polar →



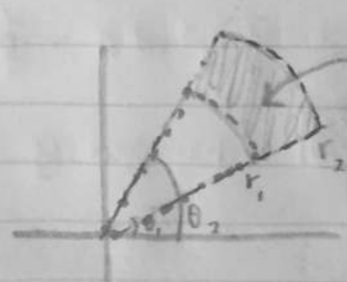
height function: $h(r\cos(\theta), r\sin(\theta)) = 2\sqrt{\alpha^2 - r^2}$

differential???

$dA_{cartesian}$ vs dA_{polar}

small changes of area in the polar system are represented by rectangles but the rectangles translate to circular sectors in the cartesian system

we need formula for dA_{cart} in terms of dA_{pol}



area of a full circular sector

$\frac{1}{2}\theta r^2 = A$

our shaded region is the difference of the areas of 2 full circular sectors

$$A = \frac{1}{2}(\theta_2 - \theta_1)r_2^2 - \frac{1}{2}(\theta_2 - \theta_1)r_1^2 = \frac{1}{2}(\theta_2 - \theta_1)(r_2^2 - r_1^2)$$

$$= \frac{1}{2}(r_1 + r_2)(\theta_2 - \theta_1)(r_2 - r_1)$$

(relating dA_{cart} and dA_{pol} continued)

$$\Delta A_{cart} = \frac{1}{2}(r_1+r_2)\Delta\theta\Delta r = \frac{1}{2}(r_1+r_2)\,\Delta A_{pol}$$

if $\Delta A_{pol} \longrightarrow 0$ $(\Delta\theta \to 0$ AND $\Delta r \longrightarrow 0)$

we see $1/2(r_1+r_2) = 1/2(2r_2 - \Delta r) = r_2 + 1/2\Delta r \rightarrow r^*$

$$\text{so, } dA_{cart} = r\,dA_{pol}$$

(back to calculating volume of a sphere using polar coordinates)

$$\mathrm{Vol}(S_\alpha) = \iint_{R_\alpha} h(x,y)\,dA_{cart} = \iint_{R_{pol}} h(r\cos\theta, r\sin\theta)\,r\,dA_{pol}$$

$$R_{pol} = \{(r,\theta) : 0 \le r \le \alpha,\ 0 \le \theta \le 2\pi\}$$

$$\int_0^{2\pi}\int_0^{\alpha} 2\sqrt{\alpha^2-r^2}\,r\,dr\,d\theta$$

$w = a^2 - r^2$

$dw = -2r\,dr$

$r\,dr = -1/2\,dw$

inner: $-\int_0^{\alpha}\sqrt{w}\,dw = -\frac{2}{3}w^{3/2}\Big|_0^{\alpha} = -\frac{2}{3}(\alpha^2-r^2)^{3/2}\Big|_0^{\alpha}$

$= -\frac{2}{3}(\alpha^2-\alpha^2)^{3/2} + \frac{2}{3}(\alpha^2-0)^{3/2}$

$= \frac{2}{3}(\alpha^2)^{3/2} = 2/3\,\alpha^3$

outer: $\frac{2}{3}\int_0^{2\pi}\alpha^3\,d\theta = \frac{2}{3}\alpha^3\theta\Big|_0^{2\pi} = \frac{2}{3}\alpha^3(2\pi) - \frac{2}{3}\alpha^3(0)$

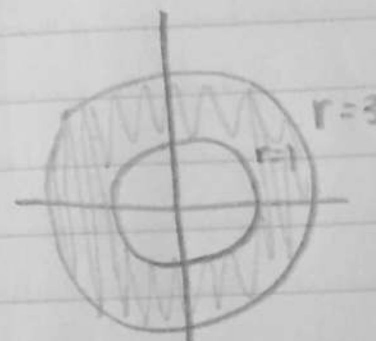
$\mathrm{Vol}(S_\alpha) = \frac{4}{3}\alpha^3\pi$ ▨ easy peasy

EX: compute the $\iint_{R_{cart}} \cos(\sqrt{x^2+y^2})\,dA_{cart}$

for R the annulus between $x^2+y^2=1$ and $x^2+y^2=9$

(annulus is the space between two circles)

graphic



$R_{pol} = \{(r,\theta) : 1 \le r \le 3,\ 0 \le \theta \le 2\pi\}$

$\cos(\sqrt{x^2+y^2}) = \cos(\sqrt{r^2}) = \cos(r) \qquad (r \ge 0)$

$x = r\cos\theta$

$y = r\sin\theta$

$\iint_{R_{cart}} \cos(\sqrt{x^2+y^2})\,dA_{cart} = \iint_{R_{pol}} \cos(r)\,r\,dA_{pol}$

$\int_1^3 \int_0^{2\pi} r\cos(r)\,d\theta\,dr$

inner: $\int_0^{2\pi} r\cos(r)\,d\theta = r\theta\cos(r)\Big|_0^{2\pi}$

$= 2\pi r\cos(r)$

outer: $\int_1^3 2\pi r\cos(r)\,dr$

$u = r \qquad dv = \cos(r)\,dr$

$du = dr \qquad v = \sin(r)$

$2\pi r\sin(r) - 2\pi\int_1^3 \sin(r)\,dr$

$2\pi r\sin(r) + 2\pi\cos(r)\Big|_1^3$

$6\pi\sin(3) + 2\pi\cos(3) - 2\pi\sin(1) - 2\pi\sin(1)$ ☐

EXERCISE: compute $\iint_R y\exp(-x^2-y^2)\,dA$ on region R the quarter annulus of the first quadrant